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Spherical Deconstruction

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Abstract

We present evidence that $\mathcal{N} = 1^*$ SUSY Yang-Mills provides a deconstruction of a six-dimensional gauge theory compactified on a two-sphere. The six-dimensional theory is a twisted compactification of $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory of the type considered by Maldacena and Nunez (MN) in [1]. In particular, we calculate the full classical spectrum of the $\mathcal{N} = 1^*$ theory with gauge group $U(N)$ in its Higgs vacuum. In the limit $N \to \infty$, we find an exact agreement with the Kaluza-Klein spectrum of the MN compactification.

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1 Introduction

One of the interesting consequences of recent developments in string theory is the possibility that consistent (non-gravitational) quantum theories exist in dimensions greater than four [2]. The idea of *deconstruction* [3] is a promising approach to defining these theories as certain limits of four-dimensional gauge theories[1]. Typically, the four-dimensional theory in its Higgs phase can be reinterpreted as a theory with additional compact, discretised directions. The hope is that higher-dimensional Lorentz invariance is recovered in an approporiate continuum limit.

In this paper we will study a version of deconstruction which leads to a six-dimensional gauge theory compactified to four-dimensions on a sphere. The starting point is $\mathcal{N} = 1^*$ SUSY Yang-Mills theory with gauge group $U(N)$. This theory contains a $U(N)$ gauge multiplet together with three adjoint chiral multiplets of equal mass M. Dependence of the superpotential on the mass parameter can be absorbed by a suitable rescaling of the fields. The superpotential then takes the form,

$$W = \mathrm{Tr}_N \left(i\Phi_1[\Phi_2, \Phi_3] + \frac{1}{2}\sum_{i=1}^{3} \Phi_i^2 \right) \tag{1}$$

The F-flatness condition is,

$$[\Phi_i, \Phi_j] = i\varepsilon_{ijk}\Phi_k \tag{2}$$

As (2) coincides with the Lie algebra of $SU(2)$, it can be solved by any N-dimensional representation of the $SU(2)$ generators [5]. This choice also solves the D-flatness condition. We will consider the vacuum corresponding to the solution $\Phi_i = J_i^{(N)}$. Here and in the following $J_i^{(n)}$ denote the generators of the unique irreducible representation of $SU(2)$ of dimension n. In this ground state, the $U(N)$ gauge group is broken to the central $U(1)$ by the Higgs mechanism. We will refer to this state as the Higgs vacuum.

The emergence of additional dimensions in this model occurs in a way which is very familiar in the context of M(atrix) theory [6, 7], and was first interpreted in terms of deconstruction in [8]. Defining rescaled fields $\hat{x}_i = 2\Phi_i/\sqrt{N^2-1}$, the expectation values in the Higgs vacuum satisfy,

$$\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = \mathbf{1} \tag{3}$$

This is the defining equation of the fuzzy two-sphere [9]. The fuzzy sphere is a discrete, non-commutative version of the usual two-sphere. As we review below, expanding the $N \times N$ matrix fields of the $\mathcal{N} = 1^*$ theory around this background naturally leads to a description in terms of fields defined on the

[1] For other relevant work see [4].

fuzzy sphere. The resulting six-dimensional theory is non-commutative and has a UV cutoff for finite N. Taking the limit $N \to \infty$ at the classical level we obtain a commutative, continuum theory on $\Re^{3,1} \times S^2$.

The appearance of a six-dimensional theory can also be understood via the string theory realisation of the $\mathcal{N} = 1^*$ theory [10]. The theory is realised on the worldvolume of N D3 branes in the presence of a background three-form field strength leading to a version of the Myers effect, where the D3 branes polarize into a spherically wrapped D5 brane [11]. The theory living on the worldvolume of the D5 brane reduces to a six-dimensional $U(1)$ gauge theory at low energies. The D3 brane charge is realised as N units of magnetic flux through the two-sphere leading to non-commutativity in the world-volume gauge theory [12] as expected.

The main aim of this paper is to understand more precisely what kind of six-dimensional theory emerges from the set-up described above. The string theory picture suggests that we should start from a $U(1)$ gauge theory in six dimensions with $\mathcal{N} = (1,1)$ supersymmetry. This is the theory living on a single D5 brane with world-volume $\mathcal{R}^{5,1}$. This theory should then be compactified on a two-sphere in a way which preserves $\mathcal{N} = 1$ supersymmetry in the four non-compact dimensions. A conventional compactification would break all the supersymmetry as there is no covariantly constant spinor on S^2. On the other hand, a *twisted* compactification of the type which occurs when a D5-brane is wrapped on a noncontractible two-cycle of a Calabi-Yau threefold is known to preserve the required fraction of the supersymmetry [13]. This compactification of the $\mathcal{N} = (1,1)$ theory, which was studied by Maldacena and Nunez (MN) in [1], is therefore a natural candidate for the theory we seek.

In this paper we will test this hypothesis by calculating the complete classical spectrum of the $\mathcal{N} = 1^*$ theory in the Higgs vacuum[2] and comparing it with the spectrum of the twisted compactification of the six-dimensional theory described above. We find that the masses and degeneracies of states in the two theories agree exactly for $N = \infty$. This result suggests that the Higgsed $\mathcal{N} = 1^*$ theory may be classically equivalent to a twisted S^2-compactification of a six-dimensional $U(1)$ gauge theory in this limit. The radius R of the two-sphere of is identified with M^{-1}, the inverse mass parameter of the four-dimensional theory. This proposal could be tested further by explicitly comparing the Lagrangians of the two theories [15]. Remarkably, the agreement of the two spectra also holds for any value of N provided we truncate the expansion of the six-dimensional fields in spherical harmonics appropriately. This truncation suggest that a six-dimensional description is valid for finite N on length-scales larger than $L_{UV} \sim R/N$.

[2]For a related calculation in the context of SUSY quantum mechanics see [14] .

Our discussion of deconstruction in this paper will be purely classical. Of course the resulting six-dimensional gauge theory is non-renormalisable and does not make sense at the quantum level without some consistent UV completion. As mentioned above, the six-dimensional theory obtained at finite N automatically has a short-distance cut-off L_{UV} of order R/N. To obtain a consistent quantum theory we should seek a continuum limit where physical scales of the six-dimensional theory are held fixed as $L_{UV} \to 0$. In particular, we should hold the radius R and the six-dimensional gauge coupling $G_6^2 \sim g^2R^2/N$ fixed[3]. This requires taking $N \to \infty$ with g^2/N held fixed, which is a strong-coupling limit of the four-dimensional theory. In [16], one of the authors studied this strongly-coupled limit for a closely related model which deconstructs a toroidal compactification of six-dimensional gauge theory. Using S-duality and the AdS/CFT correspondence the proposed continuum limit was related to the decoupling limit of the Neveu-Schwarz fivebrane which defines Little String Theory (LST). It is natural to conjecture that a similar treatment of the $\mathcal{N} = 1^*$ theory would lead to the twisted spherical compactification of LST considered in [1]. In particular, this LST reduces at low-energies to the compactified six-dimensional gauge theory we discuss in this paper. The classical results described below provide some motivation for future work along these lines.

In the next two Sections we consider in turn the six-dimensional and four-dimensional theories. Some additional information about vector spherical harmonics is relegated to an Appendix

2 Twisted Compactification

In this Section we will describe the Maldacena-Nuñez (MN) compactification of $\mathcal{N} = (1,1)$ SUSY Yang-Mills in six dimensions and find its classical spectrum of Kaluza-Klein modes. We start from the $U(1)$ theory defined on six-dimensional Minkowski space $\Re^{5,1}$. The global symmetry group is

$$SO(5,1) \times SO(4) \simeq SU(4) \times SU(2)_A \times SU(2)_B \tag{4}$$

Here $SO(5,1)$ is the six dimensional Lorentz group and $SO(4)$ is the R-symmetry of the $\mathcal{N} = (1,1)$ superalgebra. The matter content is a single $U(1)$ vector multiplet of $\mathcal{N} = (1,1)$ supersymmetry. It contains [4] a six dimensional gauge field A_M, four real scalar fields ρ_i and two Weyl spinors of opposite chirality; λ_l and $\tilde{\lambda}_{\bar{l}}$. Their transformation properties under the global symmetries are

[3] Here g^2 is the four-dimensional U(N) coupling. The low-energy effective gauge coupling of the unbroken $U(1)$ is g^2/N. The latter is then identified with the effective four-dimensional $U(1)$ coupling $G_4^2 = G_6^2/4\pi R^2$ arising from compactification of the six dimensional theory leading to the estimate of G_6^2 given in the text.

[4] The corresponding spacetime indices run over $M = 0, 1, \ldots, 5$, $i = 1, \ldots, 4$, $l = 1, \ldots, 4$, $\bar{l} = 1, \ldots, 4$

	$SU(4)$	$SU(2)_A$	$SU(2)_B$
A_M	$\mathbf{6}$	$\mathbf{1}$	$\mathbf{1}$
ρ_i	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$
λ_l	$\mathbf{4}$	$\mathbf{2}$	$\mathbf{1}$
$\tilde{\lambda}_{\bar{l}}$	$\bar{\mathbf{4}}$	$\mathbf{1}$	$\mathbf{2}$

Anticipating compactification of two spatial dimensions, we write the space-time as $\Re^{5,1} \sim \Re^{3,1} \times \Re^2$. This decomposition breaks the six-dimensional Lorentz group down to a subgroup,

$$H = SO(3,1) \times SO(2) \tag{5}$$

with covering group,

$$\bar{H} = SU(2)_L \times SU(2)_R \times U(1)_{45} \tag{6}$$

It is straightforward to decompose the six-dimensional fields into representations of H. The gauge field is written as,

$$\begin{aligned} A_M &= A_\mu \qquad \mu = 0,1,2,3 \\ &= A_a \qquad a = M-3 = 4,5 \end{aligned} \tag{7}$$

and we define the complex fields

$$n_\pm = \frac{1}{\sqrt{2}}\,(A_4 \pm iA_5) \tag{8}$$

Under the decomposition of the $SU(4)$ covering group, the six-dimensional spinors λ_l and $\tilde{\lambda}_{\bar{l}}$, transforming in the $\mathbf{4}$ and $\bar{\mathbf{4}}$, split according to,

$$\begin{aligned} \mathbf{4} &\rightarrow (\mathbf{2},\mathbf{1})^{+1} \oplus (\mathbf{1},\mathbf{2})^{-1} \\ \bar{\mathbf{4}} &\rightarrow (\mathbf{2},\mathbf{1})^{-1} \oplus (\mathbf{1},\mathbf{2})^{+1} \end{aligned}$$

under $SU(2)_L \times SU(2)_R$ where the superscript denotes $U(1)_{45}$ charge. Thus we obtain a total of four left-handed Weyl spinors λ^A_α, $\psi^{\tilde{A}}_\alpha$, where $A = 1,2$ and four right-handed spinors $\bar{\lambda}^A_{\dot{\alpha}}$ and $\bar{\psi}^{\tilde{A}}_{\dot{\alpha}}$ with $\tilde{A} = 1,2$. Here A and $\tilde{A}$ are indices of $SU(2)_L$ and $SU(2)_R$ respectively.

To summarize, the resulting bosonic fields then have quantum numbers,

	$SU(2)_L$	$SU(2)_R$	$U(1)_{45}$	$SU(2)_A$	$SU(2)_B$
A_μ	$\mathbf{2}$	$\mathbf{2}$	0	$\mathbf{1}$	$\mathbf{1}$
$n_\pm$	$\mathbf{1}$	$\mathbf{1}$	± 2	$\mathbf{1}$	$\mathbf{1}$
ρ_i	$\mathbf{1}$	$\mathbf{1}$	0	$\mathbf{2}$	$\mathbf{2}$

while the fermions transform as,

	$SU(2)_L$	$SU(2)_R$	$U(1)_{45}$	$SU(2)_A$	$SU(2)_B$
λ^A_α	$\mathbf{2}$	$\mathbf{1}$	$+1$	$\mathbf{2}$	$\mathbf{1}$
$\bar{\lambda}^A_{\dot{\alpha}}$	$\mathbf{1}$	$\mathbf{2}$	-1	$\mathbf{2}$	$\mathbf{1}$
$\psi^{\tilde{A}}_\alpha$	$\mathbf{2}$	$\mathbf{1}$	-1	$\mathbf{1}$	$\mathbf{2}$
$\bar{\psi}^{\tilde{A}}_{\dot{\alpha}}$	$\mathbf{1}$	$\mathbf{2}$	$+1$	$\mathbf{1}$	$\mathbf{2}$

We now want to compactify the theory by replacing the 45 plane by a two-sphere. In conventional compactification the couplings of fields to the curvature of the sphere are determined by their quantum numbers under $U(1)_{45}$ which corresponds to local rotations in the 45 plane. In the compactification of Maldacena and Nuñez the theory is twisted by embedding the local rotation group into the $SU(2)_A \times SU(2)_B$ R-symmetry group of the theory. To accomplish this we define Cartan subgroups $U(1)_A$ and $U(1)_B$ of $SU(2)_A$ and $SU(2)_B$ with corresponding generators Q_A and Q_B respectively[5]. We also define the diagonal subgroup $U(1)_T = D(U(1)_{45} \times U(1)_A)$ with generator $Q_T = Q_{45} + Q_A$. The vector multiplet fields then have quantum numbers,

	$U(1)_A$	$U(1)_T$
A_μ	0	0
$n_\pm$	0	± 2
ρ_i	± 1	± 1
λ^A_α	± 1	$\begin{pmatrix} +2 \\ 0 \end{pmatrix}$
$\bar{\lambda}^A_{\dot{\alpha}}$	± 1	$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$
$\psi^{\tilde{A}}_\alpha$	0	-1
$\bar{\psi}^{\tilde{A}}_{\dot{\alpha}}$	0	$+1$

We then compactify the theory with $U(1)_T$ playing the role of the local rotation group (instead of $U(1)_{45}$). We will refer to the $U(1)_T$ quantum number as T-spin and the six-dimensional fields can be split up accordingly as,

T-scalars:	$Q_T = 0$	$A_\mu,\ \lambda^{A=2}_\alpha,\ \bar{\lambda}^{A=1}_{\dot{\alpha}}$
T-spinors:	$Q_T = \pm 1$	$\psi^{\tilde{A}}_\alpha,\ \bar{\psi}^{\tilde{A}}_{\dot{\alpha}},\ \rho_i$
T-vectors:	$Q_T = \pm 2$	$n_\pm,\ \lambda^{A=1}_\alpha,\ \bar{\lambda}^{A=2}_{\dot{\alpha}}$

Correspondingly the terms scalar, spinor and vector will be reserved for describing the transformation properties of fields under the four-dimensional Lorentz group. The existence of a single Weyl spinor (of both chiralities) which is also a T-scalar guarentees the existence of a single massless fermion in four-dimensions as required by $\mathcal{N} = 1$ supersymmetry. Each six dimensional field has a kinetic term on S^2. After expanding in appropriate spherical harmonics, this kinetic term determines the masses of an infinite tower of four-dimensional fields. We now consider the Kaluza-Klein spectrum of each type of field in turn.

After integration by parts, the kinetic term for a T-scalar field φ defined on a two-sphere of unit radius can be written as,

$$S_\varphi = \int d^2\Omega\ \varphi \Delta_{S^2} \varphi \tag{9}$$

[5]These generators are normalised to take the values $Q_A = \pm 1$ on states in the fundamental representation of $SU(2)$.

where Δ_{S^2} is the scalar Laplacian on the two-sphere. To find the mass eigenstates we expand φ in terms of spherical harmonics as,

$$\varphi(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} \varphi_{lm} Y_{lm}(\theta,\phi) \tag{10}$$

where (θ,ϕ) are the standard spherical polar coordinates. Y_{lm} is an eigenfunction of the Laplacian Δ_{S^2} with eigenvalue $l(l+1)$ for integer $l \geq 0$ and for $m = -l, \ldots + l$. Thus for each T-scalar field in six dimensions we find a Kaluza-Klein tower of four dimensional fields with masses[6] $l(l+1)$ with degeneracy $(2l+1)$. According to the list of T-scalar fields given above we find a four-dimensional vector field, a left-handed Weyl spinor and a right-handed Weyl spinor at each mass level. The corresponding representations of the four-dimensional Lorentz group $SU(2)_L \times SU(2)_R$ are,

$$(\mathbf{2},\mathbf{2}) \oplus (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$$

For $l = 0$, the corresponding four-dimensional fields are massless and the spin quantum numbers match those of a single massless vector multiplet of $\mathcal{N} = 1$ supersymmetry. For $l > 1$, we find massive vector fields together with Weyl fermions. However, a massive vector multiplet of $\mathcal{N} = 1$ supersymmetry also includes scalar fields in four-dimensions. Thus, for $l > 0$, the fields descending from the T-scalars in six dimensions do not form complete multiplets of $\mathcal{N} = 1$ supersymmetry. This puzzle will be resolved below where we will find the additional states needed to form massive vector multiplets.

A two-component Dirac T-spinor Ψ defined on a two-sphere has kinetic term,

$$S_\Psi = \int d^2\omega \; i\bar{\Psi}\hat{\nabla}_{S^2}\Psi \tag{11}$$

The Dirac operator on the sphere is defined as,

$$i\hat{\nabla}_{S^2} = ie^{\mu a}\gamma_a\nabla_\mu \tag{12}$$

where $e^{\mu a}$ is the zweibein, γ_a represent the Clifford algebra and the spinor covariant derivative on S^2 is given as,

$$\nabla_\mu\Psi = \partial_\mu\Psi + \frac{i}{4}R_\mu^{ab}\sigma_{ab}\Psi \tag{13}$$

where R_{ab} is the spin-connection and σ_{ab} are the rotation generators. An explicit expression for the Dirac operator in spherical polars is given in [17] as,

$$i\hat{\nabla}_{S^2} = i\sigma_1\left(\frac{\partial}{\partial\theta} + \frac{\cot\theta}{2}\right) + \frac{i\sigma_2}{\sin\theta}\frac{\partial}{\partial\phi} \tag{14}$$

[6]The masses are dimensionless as we have set the radius of the two-sphere to be unity.

where σ_1 and σ_2 are Pauli matrices. A Dirac T-spinor field on S^2 can be expanded in terms of a complete basis formed by the eigenfunctions of the squared Dirac operator,

$$\left(-i\hat{\nabla}_{S^2}\right)^2 \Psi = \mu\Psi \tag{15}$$

The allowed eigenvalues correspond to $\mu = l^2$ for integer $l \geq 1$ with degeneracy $2l$ [17].

The fermionic T-spinor fields, $\psi_\alpha^{\tilde{A}}$ and $\bar{\psi}_{\dot{\alpha}}^{\tilde{A}}$ listed above can be combined to form two-component Dirac spinors on S^2 with kinetic terms (11) according to,

$$\rho_\alpha^{(\tilde{A})} = \begin{pmatrix} \psi_\alpha^{\tilde{A}} \\ \bar{\psi}_{\dot{\alpha}=\alpha}^{\tilde{A}} \end{pmatrix} \tag{16}$$

for $\tilde{A} = 1, 2$, $\alpha = 1, 2$. Thus we obtain four-species of Dirac spinors on the sphere. Each species yields $2l$ states of mass $\mu = l^2$ for $l \geq 1$ after expansion in terms of eigenstates of the squared Dirac operator. At each mass level we therefore find $8l$ off-shell degrees of freedom which can be recombined as $4l$ left-handed and $4l$ right-handed Weyl spinors in four dimensions. These Weyl spinors must be paired with bosonic fields to form multiplets of $\mathcal{N} = 1$ SUSY in four dimensions. The extra fields come from Kaluza-Klein reduction of the bosonic T-spinors ρ_i, $i = 1, 2, 3, 4$, which yield massive scalar fields in four dimensions. These states combine with the fermionic T-spinors to form massive chiral multiplets with Lorentz spins,

$$(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \oplus 2 \times (\mathbf{1}, \mathbf{1})$$

It remains to determine the Kaluza-Klein spectrum of the T-vector fields. As we have unbroken $\mathcal{N} = 1$ supersymmetry in the four non-compact dimensions it suffices to focus on the bosonic T-vector fields $n_\pm = (A_4 \pm iA_5)/\sqrt{2}$. The two real components A_4 and A_5 define a Maxwell gauge field a_ρ on the two-sphere. The resulting kinetic term reads,

$$S_A = \int d^2\Omega \; \frac{1}{4} F_{\rho\kappa} F^{\rho\kappa} \tag{17}$$

where $\rho, \kappa = 1, 2$ label coordinates on S^2 and $F_{\rho\kappa} = \partial_\rho a_\kappa - \partial_\kappa a_\rho$. We impose the gauge condition $\nabla^\rho a_\rho = 0$ and expand $\mathbf{a} = (a_1, a_2)$ in terms of vector spherical harmonics,

$$\mathbf{a} \;\; = \;\; \sum_{l,m} a_{lm} \mathbf{T_{lm}}$$

The gauge-fixed spherical harmonics $\mathbf{T}_{lm}$ appearing in this expansion are defined in Appendix A. This expansion diagonalises the Maxwell term as,

$$S_A = \sum_{l,m,l',m'} a_{lm}^\dagger a_{l'm'} l(l+1) \delta_{ll'} \delta_{mm'} \tag{18}$$

Thus the T-vector field $n_{\pm}$ yields a Kaluza-Klein tower of four-dimensional scalar fields of mass $l(l+1)$ with degeneracy $2l+1$ for $l \geq 1$. Notice that these fields are degenerate in mass with the four-dimensional vector fields coming from the Kaluza-Klein reduction of the T-scalars. In fact the number of scalar fields is just right to pair up with the massive vector fields to form massive vector multiplets of $\mathcal{N}=1$ SUSY in four dimensions with Lorentz spins,

$$(\mathbf{2},\mathbf{2}) \oplus 2 \times [(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})] \oplus (\mathbf{1},\mathbf{1})$$

The fermionic part of each multiplet includes two species of left- and right-handed Weyl fermions. One species comes from the KK reduction of the fermionic T-scalars $\lambda_{\alpha}^{A=2}$ and $\bar{\lambda}_{\dot{\alpha}}^{A=1}$ and the other comes from the reduction of the fermionic T-vectors $\lambda_{\alpha}^{A=1}$, $\bar{\lambda}_{\dot{\alpha}}^{A=2}$.

We summarise the complete Kaluza-Klein spectrum of the MN compactification in the table below,

T-scalar:

λ	States
$l(l+1)$	$(2l+1)\times \left\{(\mathbf{2},\mathbf{2}) \oplus (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\right\}$

$l = 0, 1, 2, \ldots$

T-spinor:

λ	States
l^2	$4l\times \left\{(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \oplus 2 \times (\mathbf{1},\mathbf{1})\right\}$

$l = 1, 2, 3, \ldots$

T-vector:

λ	States
$l(l+1)$	$(2l+1)\times \left\{(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \oplus 2 \times (\mathbf{1},\mathbf{1})\right\}$

$l = 1, 2, 3, \ldots$

We can also present the spectrum in terms of complete $\mathcal{N}=1$ multiplets. The spectrum includes a single massless $U(1)$ vector multiplet. The massive spectrum is labelled by a positive integer $l = 1, 2, \ldots$ and includes the following states,

Mass	Degeneracy	Multiplet
$l(l+1)$	$2l+1$	massive vector
l^2	$4l$	massive chiral

3 The Spectrum $\mathcal{N} = 1^*$ SUSY Yang-Mills

In this section we will compute the classical spectrum of the $\mathcal{N} = 1^*$ theory in the Higgs vacuum described above. As the theory has $\mathcal{N} = 1$ supersymmetry it suffices to consider the mass matrix of the fermions in the theory. The relevant terms in the Lagrangian are,

$$\begin{aligned} \mathcal{L}_F &= \mathrm{Tr}_N \Big\{ i\psi_i \epsilon_{ijk} [\Phi_k, \psi_j] + i\bar{\psi}_i \epsilon_{ijk} [\Phi_k^\dagger, \bar{\psi}_j] - i\lambda [\Phi_i^\dagger, \psi_i] \\ &\quad + i\psi_i [\Phi_i^\dagger, \lambda] + i\bar{\psi}_i [\Phi_i, \bar{\lambda}] - i\bar{\lambda} [\Phi_i, \bar{\psi}_i] - \psi_i \psi_i - \bar{\psi}_i \bar{\psi}_i \Big\} \end{aligned} \tag{19}$$

where λ is the gaugino and ψ_i are the superpartners of Φ_i. In the Higgs vacuum, the scalar fields take expectation values $\Phi_i = J_i^{(N)}$ for $i = 1, 2, 3$. The resulting mass terms for the fermions are written as,

$$\mathcal{L}_F = \Big\{ \big(\Psi_R\big)_{ab} \Delta^{(RS)}_{ab,cd} \big(\Psi_S^T\big)_{cd} + \big(\bar{\Psi}_R\big)_{ab} \bar{\Delta}^{(RS)}_{ab,cd} \big(\bar{\Psi}_S^T\big)_{cd} \Big\} \tag{20}$$

where the four species of Weyl fermion are combined in a column Ψ_R with $\Psi_R = \psi_i$ for $R = i = 1, 2, 3$ and $\Psi_4 = \lambda$.

Our aim is to find a suitable basis in which to diagonalise the fermion mass matrices Δ and $\bar{\Delta}$ defined above. The string theory picture of the Higgs vacuum described above suggests that the mass eigenstates should correspond to spherical harmonics on the two-sphere. As we now review, there is a natural map between $N \times N$ matrices and functions on the two-sphere which will provide the basis we seek. Scalar functions on the two sphere can be expanded as,

$$a(\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\Omega)$$

The spherical harmonics can be expressed in terms of the cartesian coordinates x_A with $A = 1, 2, 3$ of a unit vector in $\Re^3$,

$$Y_{lm}(\Omega) = \sum_{\vec{A}} f^{(lm)}_{A_1 \ldots A_l} x^{A_1} \ldots x^{A_l} \tag{21}$$

where $f^{(lm)}_{A_1 \ldots A_l}$ is a traceless symmetric tensor of $SO(3)$ with rank l. Similarly, $N \times N$ matrices can be expanded as,

$$\hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm} \tag{22}$$

$$\hat{Y}_{lm} = \sum_{\vec{A}} f^{(lm)}_{A_1 \ldots A_l} \hat{x}^{A_1} \ldots \hat{x}^{A_l} \tag{23}$$

where $\hat{x}_A = 2J_A^{(N)}/\sqrt{N^2 - 1}$. The matrices $\hat{Y}_{lm}$ are known as fuzzy spherical harmonics and they obey the orthonormality condition,

$$\mathrm{Tr}_N \left(Y^\dagger_{lm} Y_{l'm'} \right) = \delta_{l\,l'}\, \delta_{m\,m'} \tag{24}$$

The coefficient $f^{(lm)}_{A_1\dots A_l}$ is the same as on the sphere. In this way we obtain a map from $N\times N$ matrices to functions on the two-sphere;

$$\hat{a}=\sum_{l=0}^{N-1}\sum_{m=-l}^{+l}a_{lm}\hat{Y}_{lm}\rightarrow a(\Omega)=\sum_{l=0}^{N-1}\sum_{m=-l}^{l}a_{lm}Y_{lm} \tag{25}$$

Notice that the expansion in spherical harmonics is truncated at $N-1$ reflecting the finite number of degrees of freedom in the matrix $\hat{a}$.

Motivated by the above discussion we expand the fermionic fields Ψ_R in fuzzy spherical harmonics as,

$$\Psi_R=\sum_{lm}\hat{\Psi}^{(R)}_{lm}\hat{Y}_{lm} \tag{26}$$

The masses of physical states are detemined by squared mass matrix,

$$M^{(RS)}_{ab,\,ef}=\bar{\Delta}^{(RT)}_{ab,\,cd}\Delta^{(TS)}_{cd,\,ef} \tag{27}$$

In order to diagonalize this matrix we consider the bilinear form,

$$\mathcal{B}=\left(\Psi^{\dagger}_R\right)_{ab}M^{(RS)}_{ab,\,ef}\left(\Psi^T_S\right)_{ef} \tag{28}$$

The expansion (26) then yields,

$$\mathcal{B}=\sum_{l=0}^{N-1}\sum_{m=-l}^{l}\sum_{l'=0}^{N-1}\sum_{m'=-l'}^{l'}\left(\hat{\Psi}^{(R)}_{lm}\right)^{\dagger}\hat{\Psi}^{(S)}_{l'm'}N^{(RS)}_{lm,\,l'm'} \tag{29}$$

with[7]

$$N^{(RS)}_{lm,\,l'm'}=\delta_{ll'}\begin{pmatrix} J^2_{(L)}+1 & -iJ^{(L)}_3 & iJ^{(L)}_2 & 0\\ iJ^{(L)}_3 & J^2_{(L)}+1 & -iJ^{(L)}_1 & 0\\ -iJ^{(L)}_2 & iJ^{(L)}_1 & J^2_{(L)}+1 & 0\\ 0 & 0 & 0 & J^2_{(L)}\end{pmatrix}_{mm'} \tag{30}$$

with $L=2l+1$.

To complete the diagonalization, we will determine the characteristic equation of the matrix $N^{(RS)}_{lm,\,l'm'}$. Consider the $(p+q)\times(p+q)$ matrix

$$\mathcal{X}=\quad\begin{array}{cc} & \begin{array}{cc}(p) & (q)\end{array}\\ \begin{array}{c}(p)\\(q)\end{array} & \begin{pmatrix} A & B\\ C & D\end{pmatrix}\end{array} \tag{31}$$

[7] As above $J^{(L)}_i$ are the generators of $SU(2)$ in the irreducible representation of dimension L

The determinant of $\mathcal{X}$ can be evaluated using the formula [18]

$$Det(\mathcal{X}) = Det(D)Det(A - BD^{-1}C) = Det(A)Det(D - CA^{-1}B) \qquad (32)$$

Applying this formula in the present case we find,

$$Det(N - \lambda \mathbf{1}) = \prod_{l=0}^{N-1} \prod_{m=-l}^{l} (\gamma^{(l)} - 1)^{2(2l+1)} (\gamma^{(l)} + l)^{2l+3} \left(\gamma^{(l)} - (l+1)\right)^{2l-1} = 0 \qquad (33)$$

where $\gamma^{(l)} = l(l+1) + 1 - \lambda$. The roots of this equation yield the eigenvalues of the fermion mass matrix. For $l = 0$ we therefore find,

Eigenvalue	Degeneracy
0	1
1	3

while for $l = 1, 2, \ldots, N-1$ we get,

Eigenvalue	Degeneracy
l^2	$2l - 1$
$l(l+1)$	$2(2l+1)$
$(l+1)^2$	$2l + 3$

To find the complete spectrum in this case we sum over all values of l. The final result is a single zero eigenvalue and two series of eigenvalues labeled by a positive integer $k = 1, 2, \ldots N-1$,

Eigenvalue	Degeneracy
k^2	$4k$
$k(k+1)$	$2(2k+1)$

Finally we find one extra eigenvalue $\lambda = N^2$ with multiplicity $2N + 1$.

Each eigenvalue corresponds to a single left-handed Weyl fermion and its right-handed charge conjugate. As the theory has unbroken $\mathcal{N} = 1$ supersymmetry we must combine these states with bosons to form complete multiplets. The full bosonic spectrum can be calculated by similar means, but it suffices to note the following constraints. In the vacuum state we are considering, the $U(N)$ gauge group is broken down to $U(1)$. The spectrum therefore includes a single massless gauge boson and $N^2 - 1$ massive gauge bosons. The Weyl fermions must combine with these states to form $\mathcal{N} = 1$ vector multiplets. Any left over fermions must be paired with scalar fields to form chiral multiplets.

Given the above spectrum of fermions the only possibility consistent with $\mathcal{N} = 1$ SUSY is as follows. In addition to a single massless vector multiplet we have two towers of multiplets labelled by $k = 1, 2, \ldots, N-1$ as tabulated below,

Mass	Degeneracy	Multiplet
$k(k+1)$	$2k+1$	massive vector
k^2	$4k$	massive chiral

The spectrum is completed by $2N+1$ chiral multiplets of mass N^2. In the limit $N \to \infty$ this precisely matches the spectrum of the Maldacena-Nunez compactification. For finite N the spectrum of the $\mathcal{N}=1^*$ theory is a subset of that of the six-dimensional theory obtained by retaining only those states with mass less than N^2 (together with $2N+1$ chiral multiplets of mass equal to N^2).

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A Vector Harmonics

We present a brief review of vector harmonics. The eigenfunctions of the scalar Laplace operator on the sphere correspond to ordinary spherical harmonics Y_{lm}. To consider vector fields on the sphere it is not consistent to expanded the components of the vector seperately in scalar spherical harmonics. Instead a vector field defined on $\Re^3$ can be expanded in terms of the following vector harmonics [19, 20].

$$\tilde{\mathbf{a}}(\Omega) = \sum_{lm} \left(t_{lm}\tilde{\mathbf{T}}_{lm} + s_{lm}\tilde{\mathbf{S}}_{lm} + r_{lm}\tilde{\mathbf{R}}_{lm} \right) \tag{34}$$

where

$$\tilde{\mathbf{R}}_{lm} = \hat{\mathbf{r}} Y_{lm} \tag{35}$$

$$\tilde{\mathbf{T}}_{lm} = \frac{1}{\sqrt{l(l+1)}} \left[-\frac{\partial Y_{lm}}{\partial\theta}\hat{\boldsymbol{\phi}} + \csc\theta \frac{\partial Y_{lm}}{\partial\phi}\hat{\boldsymbol{\theta}} \right] \tag{36}$$

$$\tilde{\mathbf{S}}_{lm} = \frac{1}{\sqrt{l(l+1)}} \left[\frac{\partial Y_{lm}}{\partial\theta}\hat{\boldsymbol{\theta}} + \csc\theta \frac{\partial Y_{lm}}{\partial\phi}\hat{\boldsymbol{\phi}} \right] \tag{37}$$

Restricting to the two-sphere $r=1$ we find $\tilde{\mathbf{R}}_{lm} = 0$. Covariant and contravariant vectors on the two-sphere can then be defined as,

$$\tilde{V}_i = h_i V^i = h_i^{-1} V_i \tag{38}$$

where $g_{ij} = h_i^2 \delta_{ij}$. The corresponding covariant vector harmonics are

$$\mathbf{T}_{lm} = \frac{1}{\sqrt{l(l+1)}} \left[-\sin\theta \partial_\theta Y_{lm} \hat{\boldsymbol{\phi}} + \csc\theta \partial_\phi Y_{lm} \hat{\boldsymbol{\theta}} \right] \tag{39}$$

$$\mathbf{S}_{lm} = \frac{1}{\sqrt{l(l+1)}} \left[\partial_\theta Y_{lm} \hat{\boldsymbol{\theta}} + \partial_\phi Y_{lm} \hat{\boldsymbol{\phi}} \right] \tag{40}$$

The Maxwell field on a two-sphere is a vector field with the gauge invariance,

$$A_\mu \to A_\mu - \partial_\mu \chi \tag{41}$$

The gauge fields can be expanded in vector harmonics and the scalar χ can be expanded in spherical harmonics. Under a gauge transformation the components transform as,

$$\begin{aligned} A'_\theta(\theta,\phi) &= \sum_{lm} (t_{lm}\csc\theta\,\partial_\phi Y_{lm} + s_{lm}\partial_\theta Y_{lm} - \chi_{lm}\partial_\theta Y_{lm}) \\ A'_\phi(\theta,\phi) &= \sum_{lm} (t_{lm}(-\sin\theta)\partial_\theta Y_{lm} + s_{lm}\partial_\phi Y_{lm} - \chi_{lm}\partial_\phi Y_{lm}) \end{aligned}$$

It therefore follows that we can set $s_{lm} = 0$ via a gauge transformation with $\chi_{lm} = s_{lm}$. The corresponding gauge fixing condition is the generally covariant analogue of the Lorentz Gauge.

$$\begin{aligned} \nabla^a A_a &= g^{ab}\nabla_b A_a \\ &= g^{ab}\partial_b A_a - g^{ab}\Gamma^c_{ba}A_c \end{aligned} \tag{42}$$

For the sphere there are only three non-zero Christoffel symbols

$$\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \tag{43}$$

Therefore

$$\begin{aligned} \nabla^a A_a &= g^{ab}\partial_b A_a + \cot\theta A_\theta \\ &= \sum_{lm} \frac{1}{\sqrt{l(l+1)}} \Big\{ t_{lm}\Big(\partial_\theta(\csc\theta\partial_\phi Y_{lm}) + \cot\theta\csc\theta\partial_\phi Y_{lm} \\ &\quad -\csc\theta\partial_\phi\partial_\theta Y_{lm}\Big) + s_{lm}\Big(\partial_\theta\partial_\theta Y_{lm} + \cot\theta\partial_\theta Y_{lm} + \csc^2\theta\partial_\phi\partial_\phi Y_{lm}\Big)\Big\} \\ &= \sum_{lm} \frac{1}{\sqrt{l(l+1)}} s_{lm}\Delta_{S^2} Y_{lm} \end{aligned} \tag{44}$$

The gauge condition $\nabla^a A_a = 0$ thus sets $s_{lm} = 0$.

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